Massive-field approach to the scalar self force in curved spacetime

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Abstract

We derive a new regularization method for the calculation of the (massless) scalar self-force in curved spacetime. In this method, the scalar self-force is expressed in terms of the difference between two retarded scalar fields: the massless scalar field, and an auxiliary massive scalar field. This field difference combined with a certain limiting process gives the expression for the scalar self-force. This expression provides a new self-force calculation method.

I. INTRODUCTION AND SUMMARY

The motion of an electric charged point-like object in a fixed background spacetime, is affected by the coupling between the object's own charge, and the field that this charge induces. This coupling results in a *self force* (also known as the "radiation reaction force") acting on the object. At leading order, the object's acceleration due to this self force (in the absence of non-gravitational external interactions) is proportional to q^2/μ, where q and μ denotes the object's charge and mass, respectively. This leading order is obtained by treating the particle's field as a linear perturbation over a fixed curved background spacetime. Analogous to the electromagnetic self-force there are other types of self forces: a scalar self-force is induced by a scalar charge, and a gravitational self-force is induced by the object's mass (in this case the object's acceleration due to the gravitational self-force is proportional to μ).

In recent years, there has been growing interest in the calculation of the self force in curved spacetime [1]- [14]. Formal expressions of the self force have been derived: Mino, Sasaki and Tanaka [1], and independently Quinn and Wald [2], have recently obtained an expression for the gravitational self force; previously, DeWitt and Brehme [15] obtained an expression for the electromagnetic self force; and Quinn [3] recently obtained an expression for the scalar self force. In the case of a weak gravitational field, explicit self-force expressions were found by DeWitt and DeWitt [16], and by Pfenning an Poisson [4]. Analysis of the self force in curved spacetime also has a practical motivation: one possible source for LISA – the planned space-based gravitational wave detector [17], is a binary system with an extreme mass ratio, which inspirals toward coalescence. Here, the self force is required for the calculation of the accurate orbital evolution of such systems. These orbits are needed in order to design templates for the gravitational waveforms of the emitted gravitational radiation. A calculation method for the self force in such binary systems, was devised by Barack and Ori [5], this method was recently improved [6, 7], and also implemented numerically in certain cases [8, 9]. For other approaches to the self force problem see [10, 11, 13].

Previous analyses have provided several expressions for the scalar self force, which are equivalent to each other. In Quinn's derivation [3], the expression for the scalar self force is composed of two different types of terms: explicit local terms; and a non-local term, expressed as a certain integral over (the partial derivatives of) the retarded scalar Green's

function. We shall refer to these terms as the local terms, and the non-local term, respectively. Very recently, Detweiler and Whiting [14], developed a different method to express the self force in curved spacetime. In the scalar-field variant of their analysis, they showed that the scalar self force can be obtained from ψ^R, which is a certain non-retarded solution of the homogeneous scalar-field equation.

In calculating the self force acting on a point like object, one typically encounters a divergent expression, from which the finite (and correct) expression for the self force should be obtained using a certain *regularization* method. In this manuscript we present a new self-force regularization method. We consider a point-like scalar charge in curved spacetime, which induces a scalar field ϕ. We show (under certain assumptions) that the scalar self force acting on this scalar charge can be expressed in terms of two retarded scalar fields, which satisfy prescribed differential equations. These fields are: ϕ – which satisfies the inhomogenous massless scalar field equation, with a charge density ρ; and ϕ_m – which satisfies the inhomogenous *massive* scalar field equation, with exactly the same charge density ρ (here m denotes the mass of the massive scalar field). More specifically, we show that the scalar self force is given by

$$f_\mu^{self}(z_0) = q \lim_{m\to\infty} \left\{ \lim_{\delta\to 0} \Delta\phi_{,\mu}(x) + \frac{1}{2} q \big[m^2 n_\mu(z_0) + m a_\mu(z_0) \big] \right\} , \tag{1}$$

where,

$$\Delta\phi(x) \equiv \phi(x) - \phi_m(x) \ . \tag{2}$$

Here, z_0 is the self-force evaluation point on the object's world line, and x is a point near the word line, defined as follows. At z_0 we construct a unit spatial vector n^μ, which is perpendicular to the object's world line but is otherwise arbitrary (i.e. at z_0 we have $n^\mu n_\mu = 1\ ,\ n^\mu u_\mu = 0$). In the direction of this vector we construct a geodesic, which extends out an invariant length δ to the point $x(z_0, n^\mu, \delta)$; throughout this manuscript u^μ and a^μ denote the object's four-velocity and four-acceleration, at z_0, respectively.

We show below that the value of f_μ^{self} given by Eq. (1) is identical to the scalar self force obtained by Quinn [3]. Therefore, the problem of calculating the scalar self force in curved spacetime is equivalent to the problem of solving the two scalar partial differential equations for the fields ϕ and ϕ_m, and then carrying out the prescribed set of operations summarized by Eq. (1). Note that for the massive field ϕ_m, only the asymptotic behavior as $m \to \infty$ is required.

We mention here a similar method developed by Coleman [18] for the derivation of the electromagnetic self force in flat spacetime (the Abraham-Lorentz-Dirac term [19, 20, 21]). In this method, the self-force regularization is obtained by replacing the Green's function of the electromagnetic four-potential with a different (more regular) function, which depends on a certain parameter. This new function behaves like the original electromagnetic Green's function at the limit where the parameter approaches infinity. Coleman obtained this new function from the Green's function of a fictitious massive electromagnetic field. This regularization method is similar to Pauli-Villars regularization in quantum field theory [22].

The derivation of Eq. (1) is based on several properties of the fields ϕ and ϕ_m, which we now briefly summarize. Previous methods [3, 6, 7, 12] used the following decomposition of the scalar field near the object's world line: $\phi(x) = \phi_{dir}(x) + \phi_{tail}(x)$. In this decomposition the direct field $\phi_{dir}(x)$ is a field that propagates along the future light cone emanating from the object. Near the object's world line this direct field can be determined using a local analysis [12], and it diverges as $\delta \to 0$. The second field in this decomposition, the tail field $\phi_{tail}(x)$, is of nonlocal nature, i.e., its value generically depends on the entire past history of the object's world line. Moreover, the tail field is finite at the limit $\delta \to 0$, and it vanishes in flat spacetime (in 1+3 dimensions). This tail field was found to be useful in the calculation of the self force in curved spacetime [6] – the above mentioned nonlocal term of the self force, can be directly obtained from $\phi_{tail,\mu}$ near the object's world line.

In a similar manner, near the object's world line the massive field $\phi_m(x)$ can be decomposed according to $\phi_m(x) = \phi_{m(dir)}(x) + \phi_{m(tail)}(x)$. Here again, the massive direct field $\phi_{m(dir)}$ is a field that is propagated along the future light cone emanating from the object, and it diverges as $\delta \to 0$. The massive tail field $\phi_{m(tail)}$, is a field of a non-local nature, and is non-singular on the world line (for a finite value of m). However, unlike the massless tail field, this massive tail field does not vanish in flat spacetime.

The formal expressions for the two direct fields $\phi_{m(dir)}$ and ϕ_{dir} (see below) show that they are identical. This property is an important element in our construction, since it implies that by subtracting ϕ_m from ϕ [see Eq. (2)] we obtain

$$\Delta\phi(x) = \phi_{tail}(x) - \phi_{m(tail)}(x)\,, \tag{3}$$

which is finite as $\delta \to 0$, while the individual fields ϕ and ϕ_m diverge at this limit. Similarly,

the partial derivatives $\Delta\phi_{,\mu}$ are also finite at this limit. However, their values at this limit depend on the direction along which the limit $\delta \to 0$ is taken (i.e., they depend on n_μ). This directional dependency is completely annihilated by the term $(qm^2/2)n_\mu$ in Eq. (1). Therefore, the expression obtained by taking the limit $\delta \to 0$ in this equation does not depend on n_μ. To illustrate some of these properties consider the following simplest example. Consider a static unit point charge situated at $\vec{z}_0$, in a flat spacetime. In this case, the fields ϕ and ϕ_m, which satisfy equations (4) and (7), respectively, are given by

$$\phi = \phi_{dir} = \frac{1}{r} \ , \ \phi_m = \frac{1}{r} + \frac{1}{r}(-1 + e^{-mr}) \, .$$

Here $r \equiv |\vec{x} - \vec{z}_0|$. We therefore find that $\Delta\phi = -\phi_{m(tail)} = \frac{1}{r}(1 - e^{-mr})$, and the gradient of this expression in the vicinity of the particle is $\nabla(\Delta\phi) = -\frac{m^2}{2}\hat{r} + O(r)$, where $\hat{r}$ is a radial unit vector. Hence, both $\Delta\phi$ and its gradient remain finite at the limit $\delta \to 0$. Moreover, in this case the self force obviously vanishes – a result that is easily derived from Eq. (1).

The calculation of the scalar self force, naturally involves derivatives of the field – $\Delta\phi_{,\mu}$ in our construction. By virtue of (3), these derivatives will include both $\phi_{tail,\mu}$ and $\phi_{m(tail),\mu}$. As we mentioned above, the non-local part of the self force can be obtained from $\phi_{tail,\mu}$. Therefore in our method $\phi_{m(tail),\mu}$, should give rise to the (above mentioned) local terms of the self force[1]. This might look strange at first sight, because the derivatives $\phi_{m(tail),\mu}$ depend on the entire past history of the object world line. Note, however that in our method the self force is obtained by taking the limit $m \to \infty$ in Eq. (1). At this limit the contribution to Eq. (1) from the term $\phi_{m(tail),\mu}$ is a local one (see calculations below). This behavior may be traced to the properties of the massive scalar Green's function at the limit $m \to \infty$.

In Sec. II we present the massive-field approach, and show that f^{self}_μ which is given by Eq. (1) is equivalent to the standard expression for the scalar self force, given in [3].

II. THE MASSIVE FIELD APPROACH

We consider a scalar field $\phi(x)$, which satisfies the scalar field equation

$$\Box\phi = -4\pi\rho \, . \tag{4}$$

[1] We comment that there is also a certain local contribution to the self-force coming from $\phi_{tail,\mu}$; this local contribution is taken into account in the calculations below.

Here $\Box\phi \equiv \phi_{;\mu}{}^{\mu}$, and $\rho(x)$ is the charge density of the scalar object. We use the signature (-+++), and natural units where $c = 1$ throughout. Our purpose is to derive an expression for the self force on a point-like object. We therefore consider a charge density of a point particle. The particle's world line is denoted by $z(\tau)$, where τ is the particle's proper time (we allow an arbitrary acceleration – presumably caused by an arbitrary external force acting on the particle), the particle's scalar charge is denoted by q and the particle's charge density is therefore

$$\rho(x) = q\int_{-\infty}^{\infty}\frac{1}{\sqrt{-g}}\delta^4[x - z(\tau)]d\tau\,. \tag{5}$$

Here $g(x)$ is the determinant of the background metric. Next, we define the field of the scalar force to be [2]

$$F_{\mu}(x) \equiv q\phi_{,\mu}\,. \tag{6}$$

This field diverges as x approaches the particle's world line. In order to extract the self force from this singular field, we need to regularaize this expression. For this purpose, we introduce an auxiliary field $\phi_m(x)$ which satisfies the massive field (Klein-Gordon) equation[3]

$$(\Box - m^2)\phi_m = -4\pi\rho\,. \tag{7}$$

Here the charge density ρ is the same charge density introduced above [i.e., it is given by Eq. (5)]. First we consider the field's mass m to be some fixed (large) quantity. Similar to Eq. (6) we define the field of the massive scalar force to be

$$F_{(m)\mu}(x) \equiv q\phi_{m,\mu}. \tag{8}$$

Like the massless field F_{μ}, the massive field $F_{(m)\mu}$ diverges as x approaches the particle's world line. However, as we show below, their difference remains finite at this limit. We therefore construct the difference field $\Delta\phi$ given by Eq. (2), and introduce the difference between the massless and the massive scalar forces

$$\Delta F_{\mu}(x) \equiv F_{\mu}(x) - F_{(m)\mu}(x) = q\Delta\phi_{,\mu}\,. \tag{9}$$

This field is an essential element in our construction, since in our method the scalar self force is calculated from ΔF_{μ} – see Eq. (1). To analyze the properties of this field, we derive

[2] This definition conforms with definitions given in [3, 5]; a different definition is also possible [12].
[3] Note that here the field's mass m has the dimensions of $(length)^{-1}$.

a formal expression for its value near the particle's world line. This expression will later be used to obtain Eq. (1).

First, we derive a formal expression for the field $\Delta\phi$ using the corresponding retarded Green functions of the massive, and massless fields. The retarded solutions of equations (4,7) for the charge density given by Eq. (5), are formally given by

$$\phi(x) = q\int_{-\infty}^{\infty} G(x|z(\tau))d\tau, \tag{10}$$

$$\phi_m(x) = q\int_{-\infty}^{\infty} G_m(x|z(\tau))d\tau\,. \tag{11}$$

Here, $G(x|x')$ and $G_m(x|x')$ are the corresponding retarded Green's functions which satisfy

$$\Box G(x|x') = -\frac{4\pi}{\sqrt{-g}}\delta^4(x-x'),$$

$$(\Box - m^2)G_m(x|x') = -\frac{4\pi}{\sqrt{-g}}\delta^4(x-x').$$

Both Green's functions vanish for all points x which are outside the future light cone of x'. We assume that the spacetime is globally hyperbolic, and therefore G and G_m are uniquely determined by the above requirements. We further assume that the integrals in equations (10,11) converge in some local neighborhood of the particle's world line, but not on the world line itself. Expressing $\Delta\phi$ with these Green functions gives

$$\Delta\phi(x) = q\int_{-\infty}^{\infty} \Delta G(x|z(\tau))d\tau\,, \tag{12}$$

where $\Delta G(x|x') \equiv G(x|x') - G_m(x|x')$. Consider the above expression for $\Delta\phi(x)$ in some local neighborhood of the particle's world line. In the discussion below we will use two separate expressions for $\Delta G(x|z)$ in Eq. (12): a local expression – for the case where z is in a local neighborhood (defined below) of x, and a non-local expression for the case where z is outside this local neighborhood. In the next subsection we derive the local expression for ΔG.

A. Local expression for ΔG

In the vicinity of any point x' there is a local neighborhood, in which any two points can be connected by a unique geodesic within this neighborhood, see theorem 1.2.2 in [23] (this

neighborhood is sometimes called a geodesically convex domain). In this neighborhood, the massless and massive retarded Green's functions are given by [24]

$$G(x|x') = \Theta(\Sigma(x), x')[U(x|x')\delta(\sigma) - V(x|x')\Theta(-\sigma)], \tag{13}$$

$$G_m(x|x') = \Theta(\Sigma(x), x')[U(x|x')\delta(\sigma) - V_m(x|x')\Theta(-\sigma)]. \tag{14}$$

Here, $\sigma = \sigma(x|x')$ is half the square of the invariant distance measured along a geodesic connecting x and x'; σ is negative for a timelike geodesic, positive for a spacelike geodesic, and vanishes for a null geodesic; $U(x|x')$, $V(x|x')$, and $V_m(x|x')$ are certain bi-scalars (for their definitions and properties, see [23]); $\Sigma(x)$ is an arbitrary space-like hypersurface containing x; and $\Theta(\Sigma(x), x')$ equals unity if x' is in the past of $\Sigma(x)$ and vanishes otherwise.

From equations (13,14) we find that locally (i.e., in a geodesically convex domain)

$$\Delta G(x|x') = -\Delta V(x|x')\Theta(\Sigma(x), x')\Theta(-\sigma)\,, \tag{15}$$

where,

$$\Delta V(x|x') \equiv V(x|x') - V_m(x|x')\,. \tag{16}$$

Note that the *same* "direct term" $U(x|x')\delta(\sigma)$ appears in both Green's functions in equations (13) and (14), and therefore it cancels upon their subtraction.

Each of the direct fields ϕ_{dir} and $\phi_{m(dir)}$, which were mentioned in the previous section, is obtained by integrating over the corresponding direct term. These fields are therefore the same, and are given by

$$\phi_{dir}(x) = \phi_{m(dir)}(x) = q\int_{\tau^- -\varepsilon}^{\infty} \Theta(\Sigma(x), z)U(x|z)\delta(\sigma)d\tau\,. \tag{17}$$

Here τ^- denotes the retarded proper time [i.e., the proper time at the point of intersection of the past null cone of the field evaluation point x, with the particle's world line $z(\tau)$], and ε is an arbitrary small time interval. Since these direct fields are equal, they cancel each other in Eq. (2), which gives Eq. (3).

It will be useful later to have the explicit dependence of V_m on m; for this purpose we use the Hadamard expansion. It was shown by Hadamard [25] that the bi-scalars V and V_m

can be expanded as follows[4]:

$$V(x|x') = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} v_n(x|x')\,, \tag{18}$$

$$V_m(x|x') = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \tilde{v}_n(x|x')\,, \tag{19}$$

where the coefficients v_n and $\tilde{v}_n$ satisfy certain differential equations. In the case of an analytic metric, for every point x' there is a local neighborhood in which the Hadamard expansion is uniformly convergent [25] (This is also true for a class of non-analytic metrics [26].). The local neighborhood of x' in which both equations (18,19) converge uniformly is denoted by $D(x')$. Here and below we shall consider the Hadamard expansion inside $D(x')$. Expressing the coefficients $\tilde{v}_n$ in terms of the coefficients v_n, and the bi-scalar U gives [The derivation of this expression is given in Appendix A, for a different derivation, see Friedlander [23] equation (6.4.19).]

$$V_m(x|x') = mU\frac{J_1(ms)}{s} + \sum_{n=0}^{\infty} v_n J_n(ms) \left(\frac{-s}{m}\right)^n . \tag{20}$$

Here J_n denotes the Bessel function, and we introduced $s \equiv \sqrt{-2\sigma}$. Note that the bi-scalar V_m is an even function of s. We point out that the local expression for ΔG inside $D(x')$ is obtained by substituting equations (18,20) in equations (15,16).

Before continuing the detailed calculation, we briefly discuss some of the properties of V_m, and their relations to the self force in our method. In the prescription given by Eq. (1) the asymptotic expression of $\lim_{\delta\to 0} \Delta\phi_{,\mu}$, as $m \to \infty$ is required. Now, Eq. (12) implies that $\Delta\phi$ can be determined by an integral over ΔG, and by equations (15,16) the local expression of ΔG depends on V and V_m. Therefore, to obtain the asymptotic form (as $m \to \infty$) in Eq. (1), we have to study the asymptotic form of V_m (as well as the asymptotic form of the nonlocal expression of ΔG). From Eq. (20) we find that as $m \to \infty$ the Bessel functions oscillate rapidly (for $s \neq 0$); these rapid oscillations have a canceling effect upon integration with respect to s. In fact, the calculations below show that at the limit $m \to \infty$ an integral (with respect to s) over the first term in Eq. (20) behaves much like an integral over the direct term $U\delta(\sigma)$. This is an important property in our construction[5], because it explains

[4] These two expansions are defined slightly different from the various definitions in [15], [25], and [23]. The coefficients of these various definitions are related to the coefficients in equations (18,19) by multiplying by certain numbers.

[5] A similar property was used by Coleman in the calculation in flat spacetime [18].

the source of the local terms of the self force in our method. In other methods, for example in [3], the local terms of the self-force are essentially obtained from the value of $\phi_{dir,\mu}$ near the world line, by some regularization method. Here however, in the expression for $\Delta\phi$ the direct fields ϕ_{dir} and $\phi_{m(dir)}$ canceled upon subtraction. Therefore it might seem strange that we still obtain the local terms of the self-force. This is possible because at the limit $m \to \infty$ we find that the first term in Eq. (20) behaves similarly to the above mentioned direct term.

B. Calculation of ΔF_μ near the particle's world line

We now use the above local expression for ΔG, and derive an expression for the field $\Delta F_\mu(x)$ near the world line $z(\tau)$. Before proceeding, we define some notation. The particle's proper time at the self force evaluation point z_0 is denoted by τ_0; The past intersection of the boundary of $D(z_0)$ with the world line is denoted by $z(\tau_1)$. By our construction the points z_0 and any point $z(\tilde{\tau})$, such that $\tau_0 > \tilde{\tau} \geq \tau_1$, can be connected by a unique geodesic within $D(z_0)$. It may be that all these geodesics are timelike geodesics . If this requirement is not initially satisfied, we then increase the value of τ_1 such that all these geodesics with $\tau_0 > \tilde{\tau} \geq \tau_1$ will become timelike geodesics. The field evaluation point x is taken to be sufficiently close to the particle's world line (i.e. δ is sufficiently small), such that both points x and $z(\tau^-)$ are within $D(z_0)$ and $\tau^- > \tau_1$.

Using equations (12,15), we obtain

$$\Delta\phi(x) = -q\int_{\tau_1}^{\tau^-} \Delta V(x|z)\, d\tau + q\int_{-\infty}^{\tau_1} \Delta G(x|z)\, d\tau. \tag{21}$$

This field has a finite value as x approaches the world line (see Appendix C). From equations (9,21) we find that

$$\Delta F_\mu(x) = -q^2(\tau^-)_{,\mu}\left[\Delta V\right]_{\tau^-} - q^2\int_{\tau_1}^{\tau^-} \Delta V_{,\mu} d\tau + q^2\int_{-\infty}^{\tau_1} \Delta G_{,\mu} d\tau\,. \tag{22}$$

Throughout this manuscript the indices μ, ν refer to the field evaluation point (here this point is denoted by x), and the subscript τ^- indicates that the variable $z(\tau)$ inside the brackets is evaluated at the retardation point $z(\tau^-)$. Using equations (16,18,20) we obtain

$$[\Delta V(x|z)]_{\tau^-} = -\frac{1}{2}m^2[U(x|z)]_{\tau^-}\,.$$

This relation together with Eq. (22) give the following expression for ΔF_μ

$$\Delta F_\mu(x) = F^{tail}{}_\mu(x) + \frac{1}{2}q^2 m^2 [U]_{\tau^-}(\tau^-)_{,\mu} + q^2 \int_{\tau_1}^{\tau^-} V_{m,\mu} d\tau - q^2 \int_{-\infty}^{\tau_1} G_{m,\mu} d\tau \,. \tag{23}$$

Here we introduced,

$$F^{tail}{}_\mu(x) \equiv q^2 \int_{-\infty}^{\tau_1} G_{,\mu} d\tau - q^2 \int_{\tau_1}^{\tau^-} V_{,\mu} d\tau \,. \tag{24}$$

It is useful for the calculations below, to change the splitting of the integration interval in Eq. (23) into a "smooth splitting" defined as follows. We introduce auxiliary weight functions $h_1(\tau)$ and $h_2(\tau)$ (sufficiently smooth, see below), and an arbitrarily small time interval ϵ, such that $h_1(\tau) \equiv 1$ for $\tau \geq \tau_1 + \epsilon$, $h_1(\tau) \equiv 0$ for $\tau \leq \tau_1$, and $h_1(\tau)$ varies smoothly in the interval $[\tau_1, \tau_1 + \epsilon]$. Defining $h_2(\tau) \equiv 1 - h_1(\tau)$, we find from Eq. (23) that

$$\Delta F_\mu(x) = F^{tail}{}_\mu(x) + \frac{1}{2}q^2 m^2 [U]_{\tau^-}(\tau^-)_{,\mu} + q^2 \int_{\tau_1}^{\tau^-} h_1 V_{m,\mu} d\tau - q^2 \int_{-\infty}^{\tau_1+\epsilon} h_2 G_{m,\mu} d\tau \,. \tag{25}$$

Eq. (25) provides a general formal expression for the field ΔF_μ near the object's world line. Next, we shall use Eq. (25) to show that Eq. (1) is identical to the scalar self force.

C. Derivation of the self force expression

We now follow the prescription given by Eq. (1), and perform the following successive operations on the field ΔF_μ:

i Calculating the limit $\delta \to 0$ of ΔF_μ.

ii Calculating the asymptotic form of $\lim_{\delta\to 0} \Delta F_\mu$ as m approaches infinity.

These mathematical operations will be performed separately on each term in equation (25).

1. *The first term*

Consider first the limit $\delta \to 0$ of the first term $F^{tail}{}_\mu$ in Eq. (25). Since V is smooth (see theorem 4.5.1 in [23]), and $\tau^-(x) \to \tau_0$ as $\delta \to 0$, we find from Eq. (24) that

$$\lim_{\delta\to 0} F^{tail}{}_\mu(x) = q^2 \int_{-\infty}^{\tau_1} G_{,\mu}(z_0|z(\tau)) d\tau - q^2 \int_{\tau_1}^{\tau_0} V_{,\mu}(z_0|z(\tau)) d\tau \,. \tag{26}$$

This expression is equivalent to the non-local term of the scalar self-force which was found in Ref. [3] . Since, this term is independent of m, it is not affected by limit $m \to \infty$.

2. *The second term*

Consider next the second term in Eq. (25). The second term can be expressed as [6]

$$\frac{1}{2}q^2m^2(U)_{\tau^-}(\tau^-)_{,\mu} = -\frac{1}{2}q^2m^2(U)_{\tau^-}\left(\frac{\sigma_{,\mu}}{\sigma_{,\alpha}u^\alpha}\right)_{\tau^-} . \tag{27}$$

Here the index α refers to the point $z^\alpha(\tau)$ on the particle's world line, and $u^\alpha \equiv \frac{dz^\alpha}{d\tau}$. The calculation of the limit $\delta \to 0$ of this expression follows from a local expansion of the various terms in Eq. (27). The detailed calculation is given in Appendix B, there we show that

$$\lim_{\delta\to 0}\frac{1}{2}q^2m^2(U)_{\tau^-}(\tau^-)_{,\mu} = -\frac{1}{2}q^2m^2\left(u_\mu + n_\mu\right) . \tag{28}$$

3. *The third term*

Consider next the third term in Eq. (25). First, we take the limit $\delta \to 0$ of this term. Similar to Eq. (26) we find that

$$\lim_{\delta\to 0}\int_{\tau_1}^{\tau^-} h_1 V_{m,\mu}(x|z(\tau))d\tau = \int_{\tau_1}^{\tau_0} h_1 V_{m,\mu}(z_0|z(\tau))d\tau \,.$$

We now split the integration interval into two intervals: one interval is $[\tau_1+\epsilon, \tau_0]$ in which $h_1 \equiv 1$, and the the other interval is $[\tau_1, \tau_1+\epsilon]$ in which h_1 varies smoothly, this gives

$$\int_{\tau_1}^{\tau_0} h_1 V_{m,\mu}(z_0|z(\tau))d\tau = \int_{\tau_1+\epsilon}^{\tau_0} V_{m,\mu}(z_0|z(\tau))d\tau + \int_{\tau_1}^{\tau_1+\epsilon} h_1 V_{m,\mu}(z_0|z(\tau))d\tau \,. \tag{29}$$

First we focus on the first integral on the right hand side of Eq. (29). We change the integration variable to s (with z_0 fixed). Recall that for any point $z(\tilde{\tau})$ on the world line such that $\tau_1 \le \tilde{\tau} < \tau_0$, we have a unique timelike geodesic within $D(z_0)$, which connects z_0 and $z(\tilde{\tau})$. For each $z(\tilde{\tau})$ in this range we have

$$\frac{ds}{d\tau}(z_0|z(\tilde{\tau})) = s_{,\alpha}\frac{dz^\alpha}{d\tau} < 0 \,.$$

This expression is negative because $s_{,\alpha}$ is a timelike (future directed) vector at $z(\tilde{\tau})$ which is tangent to the geodesic that connects $z(\tilde{\tau})$ and z_0; and $\frac{dz^\alpha}{d\tau}$ is the four velocity vector (also future directed) at $z(\tilde{\tau})$.

[6] This expression follows from the relation between τ^- and x on the lightcone, which is given by the equation $\sigma(x|z(\tau^-)) = 0$; see for example [3].

Substituting Eq. (20) into the first integral in Eq. (29), and interchanging the order of integration and summation gives

$$\int_{\tau_1+\epsilon}^{\tau_0} V_{m,\mu}(z_0|z(\tau))d\tau = \\ \sum_{n=-1}^{\infty}(-m)^{-n}\int_{\tilde{s}_1}^{0}\left[v_{n,\mu}J_n(ms)s^n + v_n\frac{d}{ds}\left[J_n(ms)s^n\right]s_{,\mu}\right]\frac{d\tau}{ds}ds\,.$$

Here we introduced $\tilde{s}_1 \equiv s(z_0|z(\tau_1+\epsilon))$, $v_{-1}\equiv U$, and the various bi-quantities are evaluated at the points z_0 and $z(\tau)$ [e.g., $s = s(z_0|z(\tau))$, $v_n = v_n(z_0|z(\tau))$ etc ...], this notation is used in the rest of this subsection. Using integration by parts we find that

$$\int_{\tau_1+\epsilon}^{\tau_0} V_{m,\mu}(z_0|z(\tau))d\tau = \\ \sum_{n=-1}^{\infty}\int_{\tilde{s}_1}^{0} J_n(ms)\frac{(-s)^n}{m^n}\left[\frac{d\tau}{ds}\left(v_{n,\mu} - s_{,\mu}\frac{\partial v_n}{\partial s}\right) - \frac{\partial}{\partial s}\left(\frac{d\tau}{ds}s_{,\mu}\right)v_n\right]ds + \\ \left(V_m\frac{d\tau}{ds}s_{,\mu}\right)_{s=0} - \left(V_m\frac{d\tau}{ds}s_{,\mu}\right)_{s=\tilde{s}_1}\,. \tag{30}$$

Here, in the differentiation with respect to s, z_0 is fixed; whereas in the differentiation with respect to x^μ, $z^\alpha(\tau)$ is fixed. Local analysis shows that the first boundary term in Eq. (30) is given by (see Appendix B)

$$\left(V_m\frac{d\tau}{ds}s_{,\mu}\right)_{s=0} = \left(\frac{m^2}{2} - \frac{R}{12}\right)u_\mu\,. \tag{31}$$

Here R denotes the Ricci scalar[7]. We do not need to calculate the second boundary term in Eq. (30) since it cancels with another term (see below).

Next, we calculate the asymptotic form of the integral in Eq. (30) as $m\to\infty$. Consider the integral over the n'th term in this equation. To calculate the asymptotic form of this integral, we expand the term inside the square brackets in powers of s. Integration over the terms in this expansion requires one to calculate integrals of the following form:

$$\int_{\tilde{s}_1}^{0} J_n(ms)\frac{s^{n+\alpha}}{m^n}ds = \frac{1}{m^{2n+\alpha+1}}\int_{m\tilde{s}_1}^{0} J_n(y)y^{n+\alpha}dy\,. \tag{32}$$

Note that the coefficients v_n in Eq. (30) are all smooth functions (see [23], sec. 4.3), and the expansion of the expressions that do not depend on v_n, contain only positive powers of s (see Appendix B). Therefore, in Eq. (32) we need to consider only $\alpha \geq 0$. As $m\to\infty$

[7] Here the Riemann tensor is defined with the opposite sign with respect to the definition in [15].

the integral in this equation vanishes for $n \geq 0$. Therefore, only the terms with $n = -1$ contribute to Eq. (30). In this case ($n = -1$) the expansion of the term in the square brackets in Eq. (30) includes only integer powers of s (see below). From these terms only the terms with $\alpha = 0, 1$ do not vanish as $m \to \infty$. Therefore, we need to expand the $n = -1$ terms only up to the first order in s. These expansions give (see Appendix B)

$$U\frac{\partial}{\partial s}\left(s_{,\mu}\frac{d\tau}{ds}\right) = -\frac{1}{2}a_\mu + \frac{1}{3}(\dot{a}_\mu - a^2 u_\mu)s + O(s^2)\,, \tag{33}$$

$$\frac{d\tau}{ds}\left(U_{,\mu} - s_{,\mu}\frac{\partial U}{\partial s}\right) = -\frac{1}{6}\left(R_\mu{}^\nu u_\nu + R^{\eta\nu}u_\eta u_\nu u_\mu\right)s + O(s^2)\,. \tag{34}$$

Here $\dot{a}_\mu$ denotes the covariant derivative of a_μ with respect to τ. Here and below (unless explicitly indicated otherwise) the coefficients $u_\mu, a_\mu, \dot{a}_\mu, R^{\mu\nu}, R$ are evaluated at z_0. Equations (30,31,32,33,34) give

$$\begin{aligned}&\int_{\tau_1+\epsilon}^{\tau_0} V_{m,\mu}(z_0|z(\tau))d\tau \cong \\ &\frac{1}{2}m^2 u_\mu - \frac{1}{2}m a_\mu + \frac{1}{3}(\dot{a}_\mu - a^2 u_\mu) + \left(\frac{1}{6}R_\mu{}^\nu u_\nu + \frac{1}{6}R^{\eta\nu}u_\eta u_\nu u_\mu - \frac{1}{12}R u_\mu\right) \\ &-\left(V_m\frac{d\tau}{ds}s_{,\mu}\right)_{s=\tilde{s}_1}.\end{aligned} \tag{35}$$

Here, and throughout this manuscript, the symbol $\cong$ represents equality up to terms that vanish as $m \to \infty$.

We now focus on the second integral on the right hand side of Eq. (29). Using a calculation similar to the one that was performed on the first integral in Eq. (29), we find that

$$\begin{aligned}&\int_{\tau_1}^{\tau_1+\epsilon} h_1 V_{m,\mu}(z_0|z(\tau))d\tau = \\ &\sum_{n=-1}^{\infty}\int_{s_1}^{\tilde{s}_1} J_n(ms)\frac{(-s)^n}{m^n}\left[\frac{d\tau}{ds}h_1\left(v_{n,\mu} - s_{,\mu}\frac{\partial v_n}{\partial s}\right) - \frac{\partial}{\partial s}\left(h_1\frac{d\tau}{ds}s_{,\mu}\right)v_n\right]ds + \\ &\left(h_1 V_m\frac{d\tau}{ds}s_{,\mu}\right)_{s=\tilde{s}_1} - \left(V_m h_1\frac{d\tau}{ds}s_{,\mu}\right)_{s=s_1}.\end{aligned} \tag{36}$$

Here we introduced $s_1 \equiv s(z_0|z(\tau_1))$. Recall that $h_1(s_1) = 0$, and $h_1(\tilde{s}_1) = 1$. Therefore, the second boundary term in Eq. (36) vanishes, and the first boundary term is minus the boundary term in Eq. (35). Noting that asymptotically $J_n(x) = \sqrt{\frac{2}{\pi x}}\cos[x - (n + \frac{1}{2})\frac{\pi}{2}] + O(x^{-3/2})$ [27], we find that as $m \to \infty$ the integral in Eq. (36) vanishes for $n \geq 0$. For

$n = -1$ we substitute $J_{-1}(ms) = \frac{1}{m}\frac{d}{ds}J_0(ms)$ in this integral, and integrate by parts once more. Taking the limit $m \to \infty$ we find that[8]

$$\int_{\tau_1}^{\tau_1+\epsilon} h_1 V_{m,\mu}(z_0|z(\tau))d\tau \cong \left(V_m \frac{d\tau}{ds} s_{,\mu}\right)_{s=\tilde{s}_1} . \tag{37}$$

From equations (29,35,37) we obtain

$$\begin{aligned} & q^2 \int_{\tau_1}^{\tau_0} h_1 V_{m,\mu}(z_0|z(\tau))d\tau \cong \\ & q^2 \Big\{ \frac{1}{2}m^2 u_\mu - \frac{1}{2} m a_\mu + \frac{1}{3}(\dot{a}_\mu - a^2 u_\mu) \\ & + \big(\frac{1}{6} R_\mu{}^\nu u_\nu + \frac{1}{6} R^{\eta\nu} u_\eta u_\nu u_\mu - \frac{1}{12} R u_\mu \big) \Big\} . \end{aligned} \tag{38}$$

Note that the local terms of the scalar self force found in Ref. [3] are identical to the terms that are independent of m in Eq. (38).

4. The fourth term

Consider next the fourth term on the right hand side in Eq. (25). Taking the limit $\delta \to 0$ of this term gives

$$\int_{-\infty}^{\tau_1+\epsilon} h_2 G_{m,\mu}(z_0|z(\tau))d\tau . \tag{39}$$

We now discuss the limit $m \to \infty$ of this expression. Notice that in the above calculation of the third term, we found that the value of the third term at the limit $m \to \infty$ does not depend on τ_1 [see Eq. (38)]. This suggests that the entire fourth term should vanish at the limit of interest. To further support this statement we introduce the following physically motivated assumption. It is well known that in flat spacetime as $m \to \infty$ the range of the massive field interaction approaches zero - here we shall assume that this statement is also valid in curved spacetime. Therefore, the massive scalar force $F_{(m)\mu}$ for any point that is not on the world line vanishes as $m \to \infty$. We now consider a different world line $\tilde{z}(\tilde{\tau})$, which coincides with $z(\tau)$ for $\tau \leq \tau_1 + \epsilon$, and is slightly displaced with respect to $z(\tau)$ for $\tau > \tau_1 + \epsilon$, such that z_0 is *not* on $\tilde{z}(\tilde{\tau})$. The massive force field $F_{(m)\mu}(z_0)$ for the world line $\tilde{z}$ is given by

$$F_{(m)\mu}(z_0) = q^2 \partial_\mu \int_{-\infty}^{\tilde{\tau}^-} G_m(z_0|\tilde{z}(\tilde{\tau}))d\tilde{\tau}$$

[8] Here we require that $h_1(s)$ is a C^2 function.

where $\tilde{\tau}^-$ is the retarded proper time on $\tilde{z}$. Using the same smooth splitting as in Eq. (25), we find that

$$F_{(m)\mu}(z_0) = q^2 \int_{-\infty}^{\tau_1+\epsilon} h_2 G_{m,\mu} d\tau + q^2 \partial_\mu \int_{\tau_1}^{\tilde{\tau}^-} h_1 G_m d\tilde{\tau}\,. \tag{40}$$

Notice that the first term in this equation is identical to Eq. (39). By calculating the limit $m \to \infty$ of the second integral in Eq. (40) one finds that this term vanishes at this limit [9]. Therefore, Eq. (40) together with our above assumption imply that the entire forth term vanishes at the limit of interest. We verify this conclusion for the special case of a de Sitter background spacetime (see Appendix D).

5. *Result*

By substituting the above four terms in equation (25) we obtain

$$\begin{aligned}
&\lim_{\delta\to 0} \Delta F_\mu \cong F^{tail}{}_\mu(z_0) + \\
&q^2 \Bigg\{ -\frac{1}{2} m^2 n_\mu - \frac{1}{2} m a_\mu + \frac{1}{3}(\dot{a}_\mu - a^2 u_\mu) \\
&+ \left(\frac{1}{6} R_\mu{}^\nu u_\nu + \frac{1}{6} R^{\eta\nu} u_\eta u_\nu u_\mu - \frac{1}{12} R u_\mu \right) \Bigg\}.
\end{aligned} \tag{41}$$

The term in the curly brackets in Eq. (41) has a directional dependence on n_μ and it diverges as $m \to \infty$. To remove this directional dependence, and the divergence behavior as $m \to \infty$; we subtract the term $q^2(-\frac{1}{2}m^2 n_\mu - \frac{1}{2} m a_\mu)$ from both sides of this equation. We can now safely take the limit $m \to \infty$, from which we obtain Eq. (1)

$$\begin{aligned}
&f_\mu^{self}(z_0) = q \lim_{m\to\infty} \left\{ \lim_{\delta\to 0} \Delta\phi_{,\mu} + \frac{1}{2} q m^2 n_\mu + \frac{1}{2} q m a_\mu \right\} = \\
&F^{tail}{}_\mu(z_0) + q^2 \left[\frac{1}{3}(\dot{a}_\mu - a^2 u_\mu) + \left(\frac{1}{6} R_\mu{}^\nu u_\nu + \frac{1}{6} R^{\eta\nu} u_\eta u_\nu u_\mu - \frac{1}{12} R u_\mu \right) \right].
\end{aligned} \tag{42}$$

This last expression is identical to the scalar self force expression found in [3].

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[9] This calculation is based on the Hadamard expansion, and is similar to the calculation of the third term given above.

at this meeting, for discussions.

APPENDIX A: COEFFICIENTS RELATIONS

Here we express the coefficients $\tilde{v}_n(x|x')$ in terms of the coefficients $v_n(x|x')$ and the bi-scalar $U(x|x')$. From these relations we then derive Eq. (20).

We consider x to be within $D(x')$. The coefficients $v_n(x|x')$ and $\tilde{v}_n(x|x')$ satisfy the following recurrence differential equations along the geodesics which connects x and x' (see [23], Sec. 4.3):

$$\tilde{v}_0 + \left(\tilde{v}_{0;\mu} - \frac{\tilde{v}_0}{U}U_{;\mu}\right)\sigma_{;}{}^{\mu} = -\frac{1}{2}(\Box - m^2)U\,, \tag{A1}$$

$$\tilde{v}_n(n+1) + \left(\tilde{v}_{n;\mu} - \frac{\tilde{v}_n}{U}U_{;\mu}\right)\sigma_{;}{}^{\mu} = -\frac{1}{2}(\Box - m^2)\tilde{v}_{n-1}\,. \tag{A2}$$

Here the index μ refer to derivatives with respect to x^μ. The differential equations for the coefficients $v_n(x|x')$ are obtained by substituting $m=0$ in equations (A1,A2). We introduce an affine parameter r along the geodesic $x(r)$ extending from x' to x, where $x(0)=x'$ and $x(1)=x$. By substituting $\sigma_{;}{}^{\mu} = r\frac{dx^\mu}{dr}$, into equations (A1,A2), these equations can be integrated [23], which gives

$$\tilde{v}_0(x(r)|x') = -\frac{U}{2r}\int_0^r \left[\frac{(\Box - m^2)U}{U}\right]_{x(r')} dr' \tag{A3}$$

$$\tilde{v}_n(x(r)|x') = -\frac{U}{2r^{n+1}}\int_0^r r'^n \left[\frac{(\Box - m^2)\tilde{v}_{n-1}}{U}\right]_{x(r')} dr'\,. \tag{A4}$$

Similar expressions for $v_n(x(r)|x')$ are obtained by substituting $m=0$ in these equations which gives

$$v_0(x(r)|x') = -\frac{U}{2r}\int_0^r \left[\frac{\Box U}{U}\right]_{x(r')} dr' \tag{A5}$$

$$v_n(x(r)|x') = -\frac{U}{2r^{n+1}}\int_0^r r'^n \left[\frac{\Box v_{n-1}}{U}\right]_{x(r')} dr'\,. \tag{A6}$$

First, we consider $\tilde{v}_0$. From equations (A3,A5) we find that

$$\tilde{v}_0 = v_0 + \frac{1}{2}Um^2\,. \tag{A7}$$

Next we consider $\tilde{v}_1$. From equation (A5) we find that

$$r\frac{1/2\Box U - v_0}{U} = -\frac{\partial}{\partial r}\left(\frac{v_0 r^2}{U}\right)\,. \tag{A8}$$

By substituting Eq. (A7) into Eq. (A4) with $n = 1$ and using Eq. (A8), we obtain

$$\tilde{v}_1 = v_1 + \frac{v_0}{2} m^2 + \frac{U}{8} m^4 \, . \tag{A9}$$

By repeating this process of substitution of the coefficients $\tilde{v}_i$ in Eq. (A4) for $\tilde{v}_{i+1}$, we find that $\tilde{v}_n$ is given by

$$\tilde{v}_n = \sum_{k=-1}^{n} \left(\frac{m^2}{2} \right)^{n-k} \frac{v_k}{(n-k)!} \, , \tag{A10}$$

where $v_{-1} \equiv U$. The Hadamard expansion for V_m in Eq. (19) now takes the form

$$V_m(x|x') = \sum_{n=0}^{\infty} \frac{\sigma^n}{n!} \sum_{k=-1}^{n} \left(\frac{m^2}{2} \right)^{n-k} \frac{v_k}{(n-k)!} \, . \tag{A11}$$

We introduce $p = n - k$, and rearrange the order of the terms in Eq. (A11)[10] which gives

$$V_m(x|x') = \sum_{k=-1}^{\infty} \sigma^k v_k \sum_{p=0}^{\infty} \frac{(-1)^p (ms/2)^{2p}}{p!(p+k)!} = mU \frac{J_1(ms)}{s} + \sum_{k=0}^{\infty} v_k J_k(ms) \left(\frac{-s}{m} \right)^k \, . \tag{A12}$$

Here $s \equiv \sqrt{-2\sigma}$.

APPENDIX B: LOCAL ANALYSIS

Here we derive equations (33,34,31,28). First, consider the derivation of equation (33). This equation requires a local expansion of the expression $U \frac{d}{ds}(s_{,\mu} \frac{d\tau}{ds})$ in powers of s. We start by expanding $\sigma_{;\mu}(z_0|z(\tau))$ for a point $z(\tau)$ which is in the vicinity of point z_0, this gives

$$\sigma_{;\mu}(z_0|z(\tau_0 - t)) = [\sigma_{;\mu}] - t[\partial_\tau \sigma_{;\mu}] + \frac{t^2}{2}[\partial_\tau^2 \sigma_{;\mu}] - \frac{t^3}{6}[\partial_\tau^3 \sigma_{;\mu}] + O(t^4). \tag{B1}$$

Here we introduced the proper time difference $t \equiv \tau_0 - \tau$, and the following notation for the coincidence limit $[\sigma_{;\mu}] \equiv \lim_{t \to 0} \sigma_{;\mu}$. Here and below the indices μ, ν, η refer to the point $x^\mu = z_0^\mu$, and the index α refers to the point $z^\alpha(\tau_0 - t)$. Employing the coincidence limits given in [15, 28] we obtain

$$\sigma_{;\mu}(z_0|z(\tau)) = t u_\mu - \frac{t^2}{2} a_\mu + \frac{t^3}{6} \dot{a}_\mu + O(t^4) \, . \tag{B2}$$

[10] We do not discuss the question whether it is permissible to rearrange the order of the terms in this sum. We comment however that our resultant expression is equivalent to an expression for V_m obtained by Friedlander in a completely different manner, see [23] equation (6.4.19).

Here and below, unless indicated otherwise the coefficients $u_\mu, a_\mu, \dot{a}_\mu$ are evaluated at z_0. By differentiating the normalization $u^\mu(\tau)u_\mu(\tau) = -1$ twice, we find that $a^2 = -\dot{a}^\mu u_\mu$. From this relation together with $s^2 = -\sigma_{;\mu}\sigma_;{}^\mu$ [11] and Eq. (B2) we obtain

$$s(z_0|z(\tau)) = t + \frac{1}{24}a^2t^3 + O(t^4)\,. \tag{B3}$$

From this equation we find that

$$t = s - \frac{1}{24}a^2s^3 + O(s^4)\,, \tag{B4}$$

and therefore

$$\frac{d\tau}{ds} = -1 + \frac{1}{8}a^2s^2 + O(s^4)\,. \tag{B5}$$

From equations (B2,B3,B4) we obtain

$$s_{,\mu}(z_0|z(\tau)) = -s^{-1}\sigma_{;\mu} = -u_\mu + \frac{s}{2}a_\mu - s^2\left(\frac{\dot{a}_\mu}{6} - \frac{u_\mu a^2}{24}\right) + O(s^3)\,. \tag{B6}$$

A local expansion of the bi-scalar U [15] gives

$$U = 1 + \frac{1}{12}R^{\mu\nu}(z_0)\sigma_{;\mu}\sigma_{;\nu} + O(s^3)\,. \tag{B7}$$

From equations (B5,B6,B7) we obtain Eq. (33)

$$U\frac{\partial}{\partial s}(s_{,\mu}\frac{d\tau}{ds}) = -\frac{1}{2}a_\mu + \frac{s}{3}(\dot{a}_\mu - a^2u_\mu) + O(s^2)\,. \tag{B8}$$

Next, consider the derivation of equation (34). Here our purpose is to expand the expression $\frac{d\tau}{ds}\left(U_{,\mu} - s_{,\mu}\frac{\partial U}{\partial s}\right)$ in powers of s. Recall that in the differentiation with respect to s, x^μ is fixed; whereas in the differentiation with respect to x^μ, $z^\alpha(\tau)$ is fixed. Using equations (B5,B6,B7), and the above mentioned coincidence limits we obtain Eq. (34)

$$\frac{d\tau}{ds}\left(U_{,\mu} - s_{,\mu}\frac{\partial U}{\partial s}\right) = -\frac{1}{6}\left(R_\mu{}^\nu u_\nu + R^{\eta\nu}u_\eta u_\nu u_\mu\right)s + O(s^2)\,,$$

where $R^{\eta\nu}$ is evaluated at z_0. Next, consider the derivation of Eq. (31). From Eq. (20) we find that

$$(V_m)_{s=0} = \frac{m^2}{2}(U)_{s=0} + (v_0)_{s=0}\,.$$

Employing Eq. (B7), and noting that $(v_0)_{s=0} = -R/12$ [15], we obtain

$$(V_m)_{s=0} = \frac{m^2}{2} - \frac{R}{12}\,,$$

[11] For this relation see [15]. Note, that here we defined $s \equiv \sqrt{-2\sigma}$.

where R is evaluated at z_0. From this equation together with equations (B5,B6) we obtain Eq. (31)

$$\left(V_m \frac{d\tau}{ds} s_{,\mu}\right)_{s=0} = \left(\frac{m^2}{2} - \frac{R}{12}\right) u_\mu \, .$$

Next, consider the derivation of equation (28). This equation requires the calculation of the limit $\delta \to 0$ of the expression $(\tau^-)_{,\mu} = -\left[\sigma_{;\mu}(\sigma_{;\alpha}u^\alpha)^{-1}\right]_{\tau^-}$. We therefore expand the expression inside the brackets in powers of δ, keeping only the first leading order in this expansion. Consider first the term $\sigma_{;\mu}(x|z(\tau))$. We choose the points $x(\tau_0, n_\mu, \delta)$ and $z(\tau)$ to be in the vicinity of the point $z(\tau_0)$, such that δ is of the same order as t. Expanding $\sigma_{;\mu}$ we find that

$$\sigma_{;\mu}(x(\tau_0, n_\mu, \delta)|z(\tau_0 - t)) = -t\left[\frac{dz^\alpha}{d\tau}\sigma_{;\mu\alpha}\right] + \delta\left[\frac{dx^\nu}{d\delta}\sigma_{;\mu\nu}\right] + O(\delta^2)\, , \tag{B9}$$

where the square brackets denote the coincidence limit

$$[\sigma_{;\mu\nu}] \equiv \lim_{t\to 0}\lim_{\delta\to 0}\sigma_{;\mu\nu}\, .$$

Taking τ to be the retarded proper time $\tau = \tau^-$ gives (see [15])

$$t = \delta + O(\delta^2)\ ,\ (\sigma_{;\alpha}u^\alpha)_{\tau^-} = \delta + O(\delta^2)\, . \tag{B10}$$

From equations (27,B7,B9,B10) together with the above mentioned coincidence limits we obtain Eq. (28)

$$\lim_{\delta\to 0}\frac{1}{2}q^2 m^2 [U]_{\tau^-}(\tau^-)_{,\mu} = -\frac{1}{2}q^2 m^2\left(u_\mu + n_\mu\right)\, .$$

APPENDIX C: REGULARITY OF $\Delta\phi(x)$ AS $\delta \to 0$

Here we show (under certain assumptions; see below) that $\Delta\phi(x)$ remains finite as $\delta \to 0$. We consider a finite value of m; hence the bi-scalar $\Delta V = V - V_m$ is a smooth function (see theorem 4.5.1 in [23]). Therefore, the first integral in Eq. (21) is a continuous function of δ at z_0.

Consider next, the limit $\delta \to 0$ of the second integral in this equation. The non singularity of this limit follows from the following argument. First, examine $\lim_{\delta\to 0}\Delta\phi(x)$ for a particle with a different world line denoted by $z'(\tau')$, and defined such that it coincides with the original world line $z(\tau)$ for $\tau \le \tau_1$, and is slightly displaced with respect to $z(\tau)$ for $\tau > \tau_1$, such that z_0 is not on z'. As before, we assume that for the world line $z'(\tau')$ the integrals in

Eq. (10,11) converge in some local neighborhood of the $z'(\tau')$, but not on $z'(\tau')$ itself. Since z_0 is not on $z'(\tau')$, we find that for the world line $z'(\tau')$, $\Delta\phi(z_0)$ is finite. This implies that by expressing $\Delta\phi(z_0)$ [or $\lim_{\delta\to 0}\Delta\phi(x)$] by Eq. (21); the second integral in this equation is finite for the world line $z'(\tau')$, and therefore it is also finite for the original world line $z(\tau)$. We therefore conclude that $\Delta\phi(x)$ is finite at the limit $\delta \to 0$. Similarly, one can show that $\Delta F_\mu(x)$ is finite at the limit $\delta \to 0$; this limit, however, has directional irregularity (it depends on n_μ). This property is discussed in section II.

APPENDIX D: CALCULATION OF THE FOURTH TERM IN DE SITTER SPACETIME

Here, we provide an example of the calculation of the fourth term. We calculate the limit $m \to \infty$ of Eq. (39) in de Sitter spacetime. In this case the retarded Green's function of a massive scalar field is given by [29]

$$G_m(x|x') = \Theta(\Sigma(x), x')\left\{\left[\frac{\lambda s}{\sinh(\lambda s)}\right]^{3/2}\delta(\sigma) + \lambda^2 P'_\eta[\cosh(\lambda s)]\Theta(-\sigma)\right\}. \tag{D1}$$

Here $\sigma = \sigma(x'|x)$, the Riemann tensor is given by $R_{\mu\nu\eta\kappa} = \lambda^2(g_{\mu\eta}g_{\nu\kappa} - g_{\mu\kappa}g_{\nu\eta})$, P'_η is the derivative of the Legendre function, and the order η satisfies the equation $\eta(\eta+1) = 2 - \frac{m^2}{\lambda^2}$ (Note that P_η is the same for both roots of this equation.). We consider the case where $\frac{m^2}{\lambda^2} >> 2$, therefore $\eta \approx -\frac{1}{2} \pm i\frac{m}{\lambda}$. Substituting the above expression for G_m in Eq. (39) gives

$$\int_{-\infty}^{\tau_1+\epsilon} h_2 G_{m,\mu}(z_0|z(\tau))d\tau = \lambda^2 \int_{-\infty}^{\tau_1+\epsilon} h_2\partial_\mu\{P'_\eta[\cosh(\lambda s)]\}d\tau\,. \tag{D2}$$

Here, $s = s(z_0|z(\tau))$. Changing the integration variable to s, and substituting $\partial_\mu P'_\eta = s_{,\mu}\partial_s(P'_\eta)$ in Eq. (D2) and integrating by parts gives

$$\int_{-\infty}^{\tau_1+\epsilon} h_2 G_{m,\mu}(z_0|z(\tau))d\tau = -\lambda^2 \int_{\infty}^{\tilde{s}_1} \partial_s\left(h_2 s_{,\mu}\frac{d\tau}{ds}\right)P'_\eta[\cosh(\lambda s)]ds\,. \tag{D3}$$

Here $\tilde{s}_1 \equiv s(z_0|z(\tau_1+\epsilon))$. Note that both boundary terms vanished since $h_2(\tilde{s}_1) = 0$, and the asymptotic form of P_η (as $s \to \infty$) [30] implies that asymptotically $P'_\eta[\cosh(\lambda s)] \sim e^{(-3/2)\lambda s}$. We substitute

$$P'_\eta[\cosh(\lambda s)] = [\lambda\sinh(\lambda s)]^{-1}\frac{d}{ds}P_\eta$$

in Eq. (D3), and integrate by parts once more. In the resultant equation we substitute the following expression for the Legendre function P_η [30]:

$$P_\eta[\cosh(\lambda s)] = \frac{1}{\sqrt{\pi}}\left[\frac{\Gamma(\eta+1/2)}{\Gamma(\eta+1)}\frac{e^{(\eta+1)\lambda s}}{\left(e^{2\lambda s}-1\right)^{1/2}}F\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}-\eta,\left(1-e^{2\lambda s}\right)^{-1}\right)+c.c.\right],$$

where F is the hypergeometric function. Noting that asymptotically $\frac{\Gamma(z)}{\Gamma(z+1/2)} \sim z^{-1/2}$, we obtain

$$\lim_{m\to\infty}\int_{-\infty}^{\tau_1+\epsilon} h_2 G_{m,\mu}(z_0|z(\tau))d\tau \ = 0\,,$$

which conforms with our previous result.

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